FORMULA DI DE MOIVRE

Determiniamo le potenze intere dei numeri complessi:

$$z = \rho(\cos\theta + i \operatorname{sen}\theta) \qquad z^n = ? \qquad n \in \mathbb{N}$$

Dimostriamo $\boxed{z^n = \rho^n(\cos n\theta + i \operatorname{sen} n\theta)}$ (*)

per induzione:

1) La formula (*) è vera per $n = 1$:

$$z^1 = \rho^1(\cos 1\theta + i \operatorname{sen} 1\theta)$$

2) se la formula (*) è vera per n cioè:

$$z^n = \rho^n(\cos n\theta + i \operatorname{sen} n\theta)$$

3) dimostriamo (*) per $n+1$:

$$z^{n+1} = z^n \cdot z = \rho^n \overbrace{(\cos n\theta + i \operatorname{sen} n\theta)\,\rho(\cos\theta + i\operatorname{sen}\theta)}^{(\bullet)} =$$

(•) $\cos n\theta \cos\theta - \operatorname{sen} n\theta \operatorname{sen}\theta + i(\cos n\theta \operatorname{sen}\theta + \operatorname{sen} n\theta \cos\theta)$

$$= \rho^n \cdot \rho\left[\cos(n\theta + \theta) + i \operatorname{sen}(n\theta + \theta)\right] =$$

$$= \rho^{n+1}\left[\cos(n+1)\theta + i \operatorname{sen}(n+1)\theta\right] \quad \text{C.V.D.}$$

ESEMPIO

$$z^3 = (1 - \sqrt{3}\,i)^3 \qquad a = 1 \quad b = -\sqrt{3}$$

$$\rho = \sqrt{1+3} = 2$$

$$z = 1 - \sqrt{3}\,i = 2\left(\cos\left(-\frac{\pi}{3}\right) + i \operatorname{sen}\left(-\frac{\pi}{3}\right)\right)$$

$$\begin{cases} \cos\theta = \dfrac{1}{2} \\ \operatorname{sen}\theta = -\dfrac{\sqrt{3}}{2} \end{cases} \qquad \theta = -\frac{\pi}{3}$$

$$(1 - i\sqrt{3})^3 = \left[2\left(\cos\left(-\frac{\pi}{3}\right) + i\operatorname{sen}\left(-\frac{\pi}{3}\right)\right)\right]^3 = 2^3\left[\cos(-\pi) + i \operatorname{sen}(-\pi)\right] =$$

$$= 8(-1) = -8$$

RADICE n-ESIMA DI UN NUMERO COMPLESSO

Dato un numero complesso $z=\rho(\cos\theta+i\,\text{sen}\,\theta)$

si chiama RADICE n-ESIMA di z quel numero complesso w

tale che
$$\underline{w^n=z}.$$

Sia $w=\rho_1(\cos\theta_1+i\,\text{sen}\,\theta_1)$ allora

$$w^n=\rho_1^n(\cos n\theta_1+i\,\text{sen}\,n\theta_1)\quad\text{(DE MOIVRE)}$$

Siccome $w^n=z \iff \rho_1^n(\cos n\theta_1+i\,\text{sen}\,n\theta_1)=\rho(\cos\theta+i\,\text{sen}\,\theta)$

$$\iff \rho_1^n=\rho \quad e \quad n\theta_1=\theta+2k\pi$$

$$\iff \boxed{\begin{array}{l}\rho_1=\sqrt[n]{\rho}\\ \theta_1=\dfrac{\theta}{n}+\dfrac{2k\pi}{n}\end{array}}$$

$$\sqrt[n]{\rho(\cos\theta+i\,\text{sen}\,\theta)}=\sqrt[n]{\rho}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\,\text{sen}\left(\frac{\theta+2k\pi}{n}\right)\right)$$

$k=0,1,\dots,n-1$

ESEMPIO

Calcolare le radici cubiche di 1: $\sqrt[3]{1}$ $\quad a=1$, $b=0$, $\rho=1$

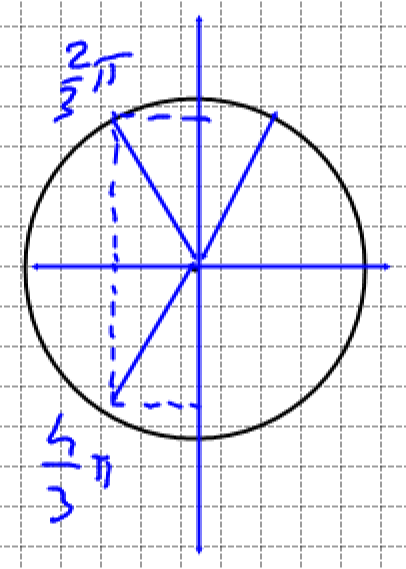
$$1=1(\cos 0+i\,\text{sen}\,0) \qquad \begin{cases}\cos\theta=1 & \theta=0+2k\pi\\ \text{sen}\,\theta=0\end{cases}$$

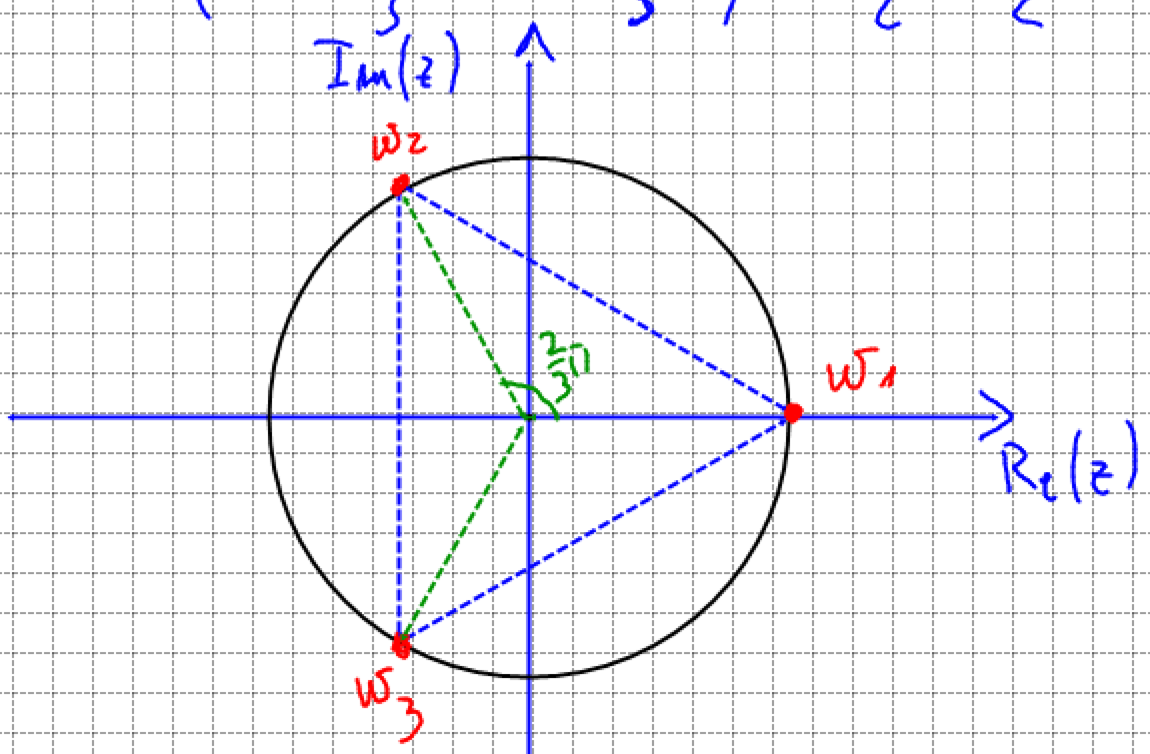
$$\sqrt[3]{1}=\sqrt[3]{1}\left(\cos\frac{0+2k\pi}{3}+i\,\text{sen}\,\frac{0+2k\pi}{3}\right) \qquad \boxed{k=0,1,2}$$

$$w_1=(\cos 0+i\,\text{sen}\,0)=1$$

$$w_2=\left(\cos\tfrac{2}{3}\pi+i\,\text{sen}\,\tfrac{2}{3}\pi\right)=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$w_3=\left(\cos\tfrac{4}{3}\pi+i\,\text{sen}\,\tfrac{4}{3}\pi\right)=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$





$$\sqrt[4]{1}=\sqrt[4]{1(\cos 0+i\,\text{sen}\,0)}=1\left[\cos\left(\frac{0+2k\pi}{4}\right)+i\,\text{sen}\left(\frac{0+2k\pi}{4}\right)\right]$$

$k=0$ $\quad w_1=1$

$k=1$ $\quad w_2=\cos\frac{\pi}{2}+i\,\text{sen}\,\frac{\pi}{2}=i$

$k=2$ $\quad w_3=\cos\pi+i\,\text{sen}\,\pi=-1$

$k=3$ $\quad w_4=\cos\frac{3}{2}\pi+i\,\text{sen}\,\frac{3}{2}\pi=-i$