PRODOTTO DI NUMERI COMPLESSI IN FORMA TRIGONOMETRICA

Dati due numeri complessi $z_1 = \rho_1(\cos\theta_1 + i\,\mathrm{sen}\,\theta_1)$,

$z_2 = \rho_2(\cos\theta_2 + i\,\mathrm{sen}\,\theta_2)$:

$$z_1 z_2 = \rho_1\rho_2(\cos\theta_1 + i\,\mathrm{sen}\,\theta_1)(\cos\theta_2 + i\,\mathrm{sen}\,\theta_2) =$$

$$= \rho_1\rho_2\Big[\underbrace{(\cos\theta_1\cos\theta_2 - \mathrm{sen}\,\theta_1\,\mathrm{sen}\,\theta_2)}_{\cos(\theta_1+\theta_2)} + i\,\underbrace{(\mathrm{sen}\,\theta_1\cos\theta_2 + \cos\theta_1\,\mathrm{sen}\,\theta_2)}_{\mathrm{sen}(\theta_1+\theta_2)}\Big] =$$

$$= \rho_1\rho_2\left[\cos(\theta_1+\theta_2) + i\,\mathrm{sen}(\theta_1+\theta_2)\right].$$

RAPPORTO

$$\frac{z_1}{z_2} = \frac{\rho_1(\cos\theta_1 + i\,\mathrm{sen}\,\theta_1)}{\rho_2(\cos\theta_2 + i\,\mathrm{sen}\,\theta_2)} = \frac{\rho_1(\cos\theta_1 + i\,\mathrm{sen}\,\theta_1)}{\rho_2(\cos\theta_2 + i\,\mathrm{sen}\,\theta_2)}\cdot\frac{\rho_2(\cos\theta_2 - i\,\mathrm{sen}\,\theta_2)}{\rho_2(\cos\theta_2 - i\,\mathrm{sen}\,\theta_2)} =$$

$$= \frac{\rho_1}{\rho_2}\,\frac{(\cos\theta_1\cos\theta_2 + \mathrm{sen}\,\theta_1\,\mathrm{sen}\,\theta_2) + i(\mathrm{sen}\,\theta_1\cos\theta_2 - \cos\theta_1\,\mathrm{sen}\,\theta_2)}{\cos^2\theta_2 + \mathrm{sen}^2\theta_2} =$$

$$= \frac{\rho_1}{\rho_2}\,\frac{\cos(\theta_1-\theta_2) + i\,\mathrm{sen}(\theta_1-\theta_2)}{1}$$

$$= \frac{\rho_1}{\rho_2}\left[\cos(\theta_1-\theta_2) + i\,\mathrm{sen}(\theta_1-\theta_2)\right]$$

FORMULA DI DE MOIVRE

Determiniamo le potenze intere dei numeri complessi:

$$z = \rho\,(\cos\theta + i \,\mathrm{sen}\,\theta) \qquad z^n = ?$$

Dimostriamo $z^n = \rho^n (\cos n\theta + i\,\mathrm{sen}\, n\theta)$ $\quad (*)$

per induzione:

1) La formula $(*)$ è vera per $n = 1$:

$$z^1 = \rho^1 (\cos 1\theta + i\,\mathrm{sen}\, 1\theta)$$

2) se la formula $(*)$ è vera per n cioè:

$$z^n = \rho^n (\cos n\theta + i\,\mathrm{sen}\, n\theta)$$

3) dimostriamo $(*)$ per $n+1$:

$$z^{n+1} = z^n \cdot z = \rho^n (\cos n\theta + i\,\mathrm{sen}\, n\theta)\,\rho(\cos\theta + i\,\mathrm{sen}\,\theta) =$$

$$= \rho^n \cdot \rho\,\left[\cos(n\theta + \theta) + i\,\mathrm{sen}(n\theta + \theta)\right] =$$

$$= \rho^{n+1}\left[\cos(n+1)\theta + i\,\mathrm{sen}(n+1)\theta\right] \qquad \text{C.V.D}$$

ESEMPIO

$$(1 - \sqrt{3}\,i)^3 \qquad a = 1 \quad b = -\sqrt{3}$$

$$\rho = \sqrt{1+3} = 2$$

$$\begin{cases} \cos\theta = \dfrac{1}{2} \\ \mathrm{sen}\,\theta = -\dfrac{\sqrt{3}}{2} \end{cases} \qquad \theta = -\frac{\pi}{3}$$

$$(1 - i\sqrt{3})^3 = \left[2\left(\cos\left(-\frac{\pi}{3}\right) + i\,\mathrm{sen}\left(-\frac{\pi}{3}\right)\right)\right]^3 = 2^3\left[\cos(-\pi) + i\,\mathrm{sen}(-\pi)\right] =$$

$$= 8(-1) = -8$$

RADICE n-ESIMA DI UN NUMERO COMPLESSO

Dato un numero complesso $z = \rho(\cos\vartheta + i \operatorname{sen}\vartheta)$

si chiama RADICE n-ESIMA di z quel numero complesso w tale che

$$\underline{w^n = z}.$$

Sia $w = \rho_1(\cos\vartheta_1 + i\operatorname{sen}\vartheta_1)$ allora

$$w^n = \rho_1^n(\cos n\vartheta_1 + i \operatorname{sen} n\vartheta_1) \quad \text{(DE MOIVRE)}$$

Siccome $w^n = z \iff \rho_1^n(\cos n\vartheta_1 + i\operatorname{sen} n\vartheta_1) = \rho(\cos\vartheta + i\operatorname{sen}\vartheta)$

$$\iff \rho_1^n = \rho \quad e \quad n\vartheta_1 = \vartheta + 2k\pi$$

$$\iff \boxed{\begin{array}{l} \rho_1 = \sqrt[n]{\rho} \\ \vartheta_1 = \dfrac{\vartheta}{n} + \dfrac{2k\pi}{n} \end{array}}$$

$$\sqrt[n]{\rho(\cos\vartheta + i\operatorname{sen}\vartheta)} = \sqrt[n]{\rho}\left(\cos\left(\frac{\vartheta + 2k\pi}{n}\right) + i\operatorname{sen}\left(\frac{\vartheta + 2k\pi}{n}\right)\right)$$

ESEMPIO

Calcolare le radici cubiche di 1:

$a = 1$
$b = 0 \quad \rho = 1$
$\cos\vartheta = 1 \quad \vartheta = 0 + 2k\pi$
$\operatorname{sen}\vartheta = 0$

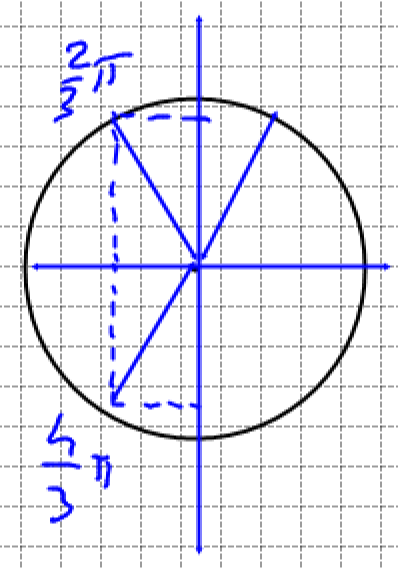
$$1 = 1(\cos 0 + i\operatorname{sen} 0)$$

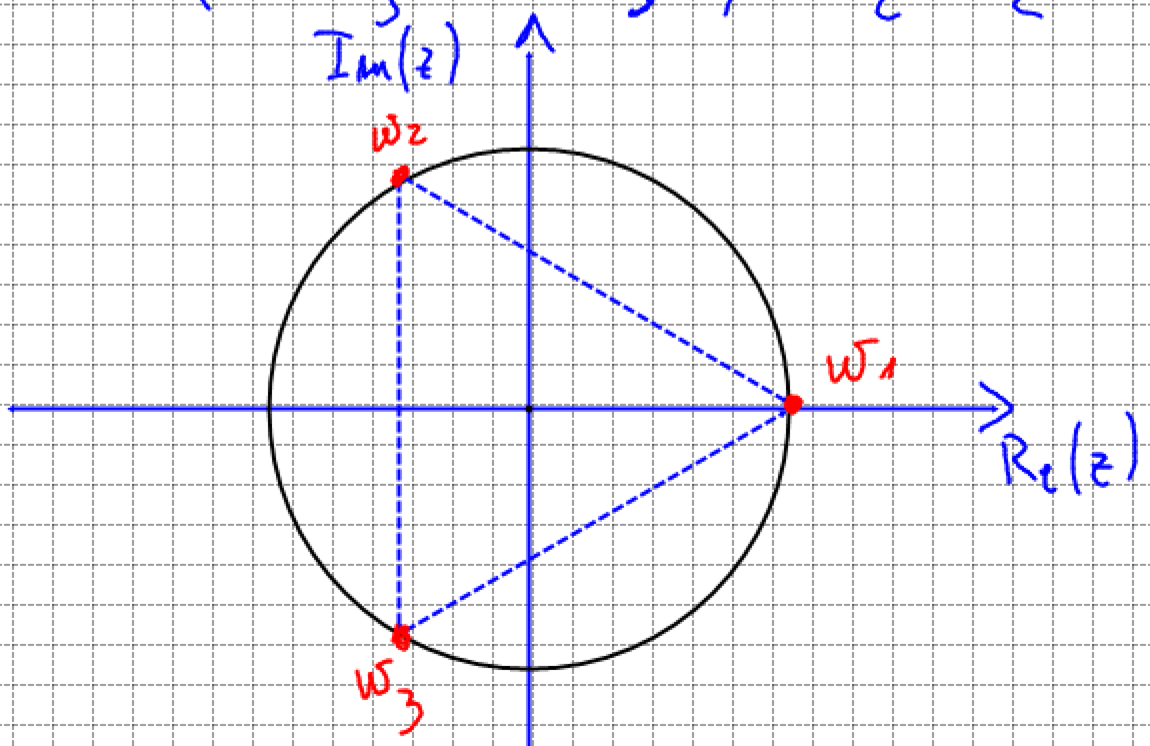
$$\sqrt[3]{1} = \sqrt[3]{1}\left(\cos\frac{0 + 2k\pi}{3} + i\operatorname{sen}\frac{0+2k\pi}{3}\right) \qquad k = 0, 1, 2$$

$$w_1 = (\cos 0 + i\operatorname{sen} 0) = 1$$

$$w_2 = \left(\cos\frac{2}{3}\pi + i\operatorname{sen}\frac{2}{3}\pi\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$w_3 = \left(\cos\frac{4}{3}\pi + i\operatorname{sen}\frac{4}{3}\pi\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$





$$\sqrt[4]{1} = \sqrt[4]{1(\cos 0 + i\operatorname{sen} 0)} = 1\left[\cos\left(\frac{0+2k\pi}{4}\right) + i\operatorname{sen}\left(\frac{0+2k\pi}{4}\right)\right]$$

$k=0 \qquad w_1 = 1$

$k=1 \qquad w_2 = \cos\frac{\pi}{2} + i\operatorname{sen}\frac{\pi}{2} = i$

$k=2 \qquad w_3 = \cos\pi + i\operatorname{sen}\pi = -1$

$k=3 \qquad w_4 = \cos\frac{3}{2}\pi + i\operatorname{sen}\frac{3}{2}\pi = -i$